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Linear Double Universal Grassmann Manifold Method for the Stationary Axisymmetric Vacuum Gravitational Field Equations

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Nagatomo's universal Grassmann manifold scheme is extended to a double form, which is used to find the exact solutions of the stationary axisymmetric vacuum gravitational field equations. Some new results are given.

1. INTRODUCTION

The stationary axisymmetric vacuum gravitational field (SAVGF) equations have been reduced to the Ernst equation (Ernst, 1968) and studied extensively. In order to obtain exact solutions of the SAVGF equations, various methods have been used, such as Backlund transformations (Harrison, 1978; Neugebauer, 1979) and inverse scattering methods (Belinskii and Zakharov, 1979; Hauser and Ernst, 1981). Several years ago, Zhong (1985) established a double complex function method and used it to study the SAVGF equations; some further new results were given by Zhong (1988) and Gao and Zhong (1992). Nagatomo (1989) pointed out that the SAVGF equations can be linearized by the use of universal Grassmann manifold (UGM) techniques. As a result, some classes of exact solutions of SAVGF equations with given initial data can be obtained. However, in the scheme of Nagatomo (1989), only ordinary complex numbers are used, and therefore, according to the theory given by Zhong (1985, 1988), the double duality symmetries must be lost. In this paper, we extend Nagatomo's UGM technique (Nagatomo, 1989) to a double form. We find that at least half of the exact solutions obtained by using our scheme cannot be obtained by using the original technique.

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In Section 2, we briefly write out some relevant notations and results of the double complex function method (Zhong, 1985). In Section 3, Nagatomo's UGM technique is extended to a double form and some new results are given. Several concrete applications of the double UGM method are given in Section 4.

2. SOME NOTATIONS AND RESULTS OF THE DOUBLE COMPLEX FUNCTION METHOD

Let J denote the double imaginary unit, i.e., $J = i$ ($i^2 = -1$) or $J = \epsilon$ ($\epsilon^2 = +1$, $\epsilon \neq \pm 1$). If the real series $\Sigma\, a_n$ is absolutely convergent, then $a(J) = \Sigma_{n=0}^{\infty} a_n J^{2n}$ is called a double real number. If $a(J)$ and $b(J)$ both are double real numbers, then $Z(J) = a(J) + J\cdot b(J)$ is called a double complex number and denoted by $Z_C = Z(J = i)$, $Z_H = Z(J = \epsilon)$.

The line element of the SAVGF can be written as

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] - f(dt + \omega\, d\phi)^2 \tag{2.1}$$

where f, ω, and γ are real functions of ρ and z only, and γ is determined by f and ω. According to Zhong (1985), Einstein's field equations can be reduced to the following double complex Ernst equation:

$$\mathrm{Re}(\mathscr{E}(J))\, \nabla^2 \mathscr{E}(J) = \nabla\mathscr{E}(J)\cdot\nabla\mathscr{E}(J) \tag{2.2}$$

where $\mathscr{E}(J) = F(J) + J\cdot\Omega(J)$ is a double complex Ernst potential, while $F(J) = F(\rho, z; J)$ and $\Omega(J) = \Omega(\rho, z; J)$ are double real functions of ρ and z. If $\mathscr{E}(J)$ is a double solution of equation (2.2), then a pair of dual solutions (f, ω) and $(\hat{f}, \hat{\omega})$ of the SAVGF equations can be obtained as

$$\begin{aligned} f &= F_{\mathrm{C}}, \qquad & \omega &= V_{F_{\mathrm{C}}}(\Omega_{\mathrm{C}}) \\ \hat{f} &= T(F_{\mathrm{H}}), \qquad & \hat{\omega} &= \Omega_{\mathrm{H}} \end{aligned} \tag{2.3}$$

where the NK transformations (T, V) are defined by

$$\begin{aligned} T&: \quad F \to T(F) = \rho/F \\ V&: \quad F, \Omega \to V_F(\Omega) = \int \frac{\rho}{F^2}(\partial_z\Omega\cdot d\rho - \partial_\rho\Omega\cdot dz) \end{aligned} \tag{2.4}$$

Let $M(J)$ be a 2×2 double real function matrix,

$$M(J) = \frac{1}{F(J)}\begin{pmatrix} 1 & \Omega(J) \\ \Omega(J) & \Omega^2(J) - J^2F^2(J) \end{pmatrix} \tag{2.5}$$

Then equation (2.2) can be written as the double BZ equations (Belinskii and Zakharov, 1979; Zhong, 1988):

$$\partial_\rho[\rho\partial_\rho M(J)\cdot M^{-1}(J)] + \partial_z[\rho\partial_z M(J)\cdot M^{-1}(J)] = 0 \tag{2.6a}$$

$$\det M(J) = -J^2, \qquad M^{\mathrm{T}}(J) = M(J) \tag{2.6b}$$

where superscript T denotes the transposition. Conversely, if $M(J)$ is a solution of equation (2.6), then

$$\mathscr{E}(J) = 1/[M(J)]_{11} + J\cdot[M(J)]_{12}/[M(J)]_{11} \tag{2.7}$$

satisfies equation (2.2).

3. DOUBLE UGM METHOD FOR SOLVING SAVGF EQUATIONS

In this section, we extend the UGM method given by Nagatomo (1989) to a double form and study its effects. For double results that can be proved by using methods similar to those of Nagatomo (1989), we shall directly write them out.

Let $\mathscr{C}[J]$ be the set of all 2×2 matrices with elements in DR (double real numbers), let $\mathscr{C}[z; J]$ and $\mathscr{C}[\rho, z; J]$ denote, respectively, the sets of all formal power series in z and (ρ, z) both with coefficients in $\mathscr{C}[J]$, and let $\mathscr{C}[z; J]^{\times}$ denote the set of invertible elements in $\mathscr{C}[z; J]$.

Theorem 1. The initial value problem

$$\partial_\rho[\rho\partial_\rho M(J)\cdot M^{-1}(J) + \partial_z[\rho\partial_z M(J)\cdot M^{-1}(J)] = 0, \quad M(J) \in \mathscr{C}[\rho, z; J] \tag{3.1a}$$

$$M(\rho, z; J)\big|_{\rho=0} = M(z; J), \quad M(z; J) \in \mathscr{C}[z; J]^{\times} \tag{3.1b}$$

has a unique double solution. If an initial datum $M(z; J)$ satisfies the supplementary conditions

$$\det M(z; J) = -J^2, \qquad M^{\mathrm{T}}(z; J) = M(z; J) \tag{3.2}$$

then the unique solution of equations (3.1) also satisfies the condition (2.6b). ■

Theorem 2. If $M(\rho, z; J)$ is a double solution of equations (3.1), then we have

$$M(-\rho, z; J) = M(\rho, z; J) \tag{3.3}$$

i.e., $M(\rho, z; J)$ is an even function of ρ.

Proof. If $M(\rho, z; J)$ is a solution of equation (3.1a), so is $M(-\rho, x; J)$. Furthermore, because both $M(\rho, z; J)$ and $M(-\rho, z; J)$ have the same initial datum at $\rho = 0$, then by Theorem 1, we have (3.3). ■

Equation (3.1a) is the integrability condition of the following double linear system (Belinskii and Zakharov, 1979; Zhong, 1988; Nagatomo, 1989):

$$D_\rho \Psi(\lambda; J) = Q(J)\Psi(\lambda; J)$$
$$D_z \Psi(\lambda; J) = P(J)\Psi(\lambda; J) \tag{3.4}$$

where

$$D_\rho \equiv \partial_\rho + \lambda\rho\partial_z, \qquad D_z \equiv \partial_z - \lambda\rho\partial_\rho + 2\lambda^2\partial_\lambda$$
$$Q(J) \equiv \partial_\rho M(J)\cdot M^{-1}(J), \qquad P(J) \equiv \partial_z M(J)\cdot M^{-1}(J) \tag{3.5}$$

and $\Psi(\lambda; J) = \Psi(\rho, z, \lambda; J)$, called a double wave function, is a 2×2 double ordinary complex matrix function of ρ, z, and the spectral parameter λ. In this paper we deal with double wave functions that are analytic at $(\rho, z; \lambda) = (0, 0; \infty)$.

Theorem 3. Let $M(\rho, z; J)$ be a solution of equation (2.6). Then there exists a unique double wave function

$$\Psi(\rho, z, \lambda; J) = I_2 + \sum_{j=1}^{\infty} \psi_j(\rho, z; J)\lambda^{-j}, \qquad \psi_j(\rho, z; J) \in \mathscr{C}[\rho, z; J]$$

such that

$$D_\rho \Psi(\lambda; J) = Q(J)\Psi(\lambda; J), \qquad D_z \Psi(\lambda; J) = P(J)\Psi(\lambda; J)$$
$$\Psi(\rho, z, \lambda; J)\big|_{\lambda=\infty} = I_2 \quad (2 \times 2 \text{ unit matrix}) \tag{3.6}$$

■

Furthermore, we have the following result.

Theorem 4. Equations (3.6) are equivalent to the following initial value problem:

$$D_\rho \Psi(\lambda; J) = Q(J)\Psi(\lambda; J)$$

$$\Psi(\rho, z, \lambda; J) = I_2 + \sum_{j=1}^{\infty} \psi_j(\rho, z; J)\lambda^{-j}, \qquad \psi_j(\rho, z; J) \in \mathscr{C}[\rho, z; J]$$

$$\Psi(\rho, z, \lambda; J)\big|_{\rho=0} = M(z; J)\cdot\left[M\left(z + \frac{1}{2\lambda}; J\right)\right]^{-1} \tag{3.7}$$

In terms of the double function set $\{\psi_j(\rho, z; J)\}$, equations (3.7) can be written as

$$\rho\partial_z\psi_j(J) = -\partial_\rho\psi_{j-1}(J) + Q(J)\psi_{j-1}(J)$$
$$\psi_j(\rho, z; J)\big|_{\rho=0} = \psi_j(z; J) \tag{3.8}$$

where the $\psi_j(z; J)$ are determined by

$$M(z; J)\cdot\left[M\left(z + \frac{1}{2\lambda i}J\right)\right]^{-1} = \sum_{j=0}^{\infty} \psi_j(z; J)\lambda^{-j} \tag{3.9}$$

Upon taking $j = 1$ in equations (3.8) and noting that $\psi_0(J) = I_2$, we have

$$Q(J) = \rho\partial_z\psi_1(\rho, z; J) \tag{3.10}$$

The relation between the double wave function $\Psi(\rho, z, \lambda; J)$ and the solutions of equations (3.1) is given by the following result.

Theorem 5. The unique double solution $M(\rho, z; J)$ of the initial value problem

$$\partial_\rho M(\rho, z; J) = Q(\rho, z; J)M(\rho, z; J), \qquad M(\rho, z; J)\big|_{\rho=0} = M(z; J) \tag{3.11}$$

satisfies equations (3.1) [or equations (2.6) if a suitable initial datum is taken], where $Q(J)$ is given by equation (3.10).

From Theorem 5, the problem of generating solutions of equations (2.6) is reduced to that of finding $\psi_j(\rho, z; J) \in \mathscr{C}[\rho, z; J]$ which obey equations (3.8). Equations (3.8) can be linearized by the use of the following so-called double UGM method. Similar to Nagatomo (1989), we introduce an $\infty \times \infty$ double matrix function $\xi(J) = (\xi_{ij}(J))_{i\in Z, j<0}$, $\xi_{ij}(J) \in \mathscr{C}[\rho, z; J]$ obeying

$$\Lambda\xi(J) = \xi(J)C(J), \qquad \xi_{ij}(J) = \delta_{ij}I_2, \qquad \text{for} \quad i, j < 0 \tag{3.12}$$

where

$$\Lambda \equiv (\delta_{i+1,j}I_2)_{i,j\in Z} \qquad \text{and} \qquad C(J) \equiv \begin{pmatrix} (\delta_{i+1,j}I_2)_{i<-1,j<0} \\ (\xi_{0j}(J))_{j<0} \end{pmatrix}$$

Then we have a bijection between $\Psi(\rho, z, \lambda; J)$ and $\xi(\rho, z; J)$ characterized by

$$\xi_{0j}(J) = -\Psi_{-j}(J) \qquad \text{for} \quad j < 0 \tag{3.13}$$

or more explicitly

$$\xi(J) = (\psi^*_{i-j}(J))_{i\in Z, j<0}\cdot(\psi_{i-j}(J))_{i,j<0} \tag{3.14}$$

where we set $\psi_j(J) = \psi_j^*(J) = 0$ for $j < 0$ and the $\psi_j^*(J)$ denote the coefficients of $\Psi^{-1}(\rho, z, \lambda; J)$ for $j \geq 0$, i.e.,

$$\Psi^{-1}(\rho, z, \lambda; J) = \sum_{j=0}^{\infty} \psi_j^*(J)\lambda^{-j}$$

By using the matrix $\xi(J)$, we can write equations (3.8) as

$$\rho\partial_z\xi_{i+1,j}(J) + \partial_\rho\xi_{ij}(J) - \xi_{i,-1}(J)\rho\partial_z\xi_{0j}(J) = 0 \tag{3.15}$$

for $i \in \mathbf{Z}, j < 0$. A linearization of this equation is given by the following result.

Theorem 6. Let $\xi^{(0)}(z; J)$ be an $\infty \times \infty$ matrix satisfying equations (3.12) and $\xi^{(0)}_{0j}(z; J) = -\psi_{-j}(z; J), j < 0$. Define the $\infty \times \infty$ matrices

$$\tilde{\xi}(\rho, z; J) = \exp\left(-\frac{1}{2}\rho^2\Lambda\partial_z\right)\xi^{(0)}(z; J)$$

$$= \sum_{k=0}^{\infty}\left(-\frac{1}{2}\rho^2\Lambda\partial_z\right)^k\xi^{(0)}(z; J)/k!$$

$$\tilde{\xi}_{(-)}(\rho, z; J) = (\tilde{\xi}_{ij}(\rho, z; J))_{i,j<0} \tag{3.16}$$

Then the inverse $\tilde{\xi}^{-1}_{(-)}(J)$ and the product $\tilde{\xi}(J)\cdot\tilde{\xi}^{-1}_{(-)}(J)$ both can be defined as $\infty \times \infty$ matrices, and the double matrix $\xi(J) = \tilde{\xi}(J)\cdot\tilde{\xi}^{-1}_{(-)}(J)$ satisfies equations (3.15) and (3.12).

By using the above theorems, we can construct some exact solutions of the SAVGF equations. Noting that the formulas in this paper all are double, from each solution $M(\rho, z; J)$ of equations (3.1) we can obtain a pair of dual SAVGF solutions (f, ω) and $(\hat{f}, \hat{\omega})$ by equations (2.7) and (2.3). Let S and $\hat{S}$ denote the sets consisting of the solutions $\{(f, \omega)\}$ and $\{(\hat{f}, \hat{\omega})\}$, respectively; then we have the following result.

Theorem 7. $S \cap \hat{S} = \varnothing$ (empty set).

Proof. According to Theorem 2, $M(\rho, z; J)$ is an even function of ρ, and from equations (2.7) and (2.3), $f = 1/M_{11}(J = i), \hat{f} = \rho M_{11}(J = \epsilon)$; thus f is an even and $\hat{f}$ is an odd function of ρ, and we have $\{f\} \cap \{\hat{f}\} = \varnothing$, which means that equation (3.17) holds. ∎

Remark. Theorem 7 is proved for the case of formal series solutions; however, in the general case, it does not necessarily hold.

The set S mentioned above is evidently the set of all solutions that can be obtained by Nagatomo's UGM method (Nagatomo, 1989). In this sense, Theorem 6 shows that the solutions in set $\hat{S}$ are indeed new. Furthermore, since there is a bijection between the formal power series solutions $M(\rho, z; J)$ and the initial data $M(z; J)$ and the latter must satisfy the conditions (3.2), for each solution $(f, \omega) \in S$ there exists a dual solution $(\hat{f}, \hat{\omega}) \in \hat{S}$. On the other hand, there may be no dual solutions for some solutions in $\hat{S}$ (see the next section). Thus we can conclude that by using the above double UGM method, we can obtain solutions which at least double in number those that can be obtained by using the original scheme.

4. APPLICATIONS

From the above discussions, once an initial datum $M(z; J)$ which satisfies the conditions (3.2) is given, then $\xi^{(0)}(z; J)$ is obtained by

$$\xi^{(0)}(z; J) = (\psi^*_{i-j}(z; J))_{i\in Z, j<0} \cdot (\psi_{i-j}(z; J))_{i,j<0} \tag{4.1}$$

where $\psi_j(z; J)$ is determined by equation (3.9). Then Theorem 6 is used and the corresponding double solution $M(\rho, z; J)$ of equation (2.6) is obtained by solving the following initial value problem:

$$\partial_\rho M(J) = -\rho\partial_z\xi_{0,-1}(J)\cdot M(J), \qquad M(J)\big|_{\rho=0} = M(z; J) \tag{4.2}$$

If $M(z; J)$ is a finite-degree polynomial of z, then the calculations involve essentially only finite-dimensional matrices. In the following, we give some concrete examples:

Example 1. Taking the initial datum as

$$M_1(z; J) = \begin{pmatrix} 1 & z \\ z & z^2 - J^2 \end{pmatrix} \tag{4.3}$$

which satisfies condition (3.2), then from equation (4.1) and Theorem 6 we find

$$\partial_z\xi_{0,-1}(\rho, z; J) = \frac{1}{(J^2 + \rho^2/4)} \begin{pmatrix} 1/2 & 0 \\ z & 1/2 \end{pmatrix} \tag{4.4}$$

and the corresponding double solution of equation (4.2) is

$$M_1(\rho, z; J) = \frac{1}{(1 + J^2\,\rho^2/4)} \begin{pmatrix} 1 & z \\ z & z^2 - J^2(1 + J^2\,\rho^2/4)^2 \end{pmatrix} \tag{4.5}$$

By equation (2.7) we obtain a double complex Ernst potential

$$\mathscr{E}_1(J) = \left(1 + J^2\frac{\rho^2}{4}\right) + Jz \tag{4.6}$$

and equations (2.3) give two SAVGF solutions

$$(f_1, \omega_1) = (1 - \rho^2/4,\ 2/(1 - \rho^2/4)) \tag{4.7a}$$

$$(\hat{f}_1, \hat{\omega}_1) = (\rho/(1 + \rho^2/4),\ z) \tag{4.7b}$$

By using the results of Nakamura (1983) and Zhong (1985), we can take the following transformation of the double complex Ernst potential:

$$\mathscr{E}(J) \to \tilde{\mathscr{E}}(J) = \mathscr{E}^{-1}(J) \tag{4.8}$$

Then we obtain another Ernst potential

$$\mathscr{E}_1(J) = \frac{1 + J^2 \rho^2/4}{(1 + J^2 \rho^2/4)^2 - J^2z^2} - J\cdot\frac{z}{(1 + J^2 \rho^2/4)^2 - J^2z^2} \tag{4.9}$$

and from equations (2.3) another pair of SAVGF solutions are given as

$$(f_1', \omega_1') = \left(\frac{1 - \rho^2/4}{(1 - \rho^2/4)^2 + z^2}, \frac{\rho^2(1 - \rho^2/4 - z^2)}{2(1 - \rho^2/4)}\right) \tag{4.10a}$$

$$(\hat{f}_1', \hat{\omega}_1') = \left(\frac{\rho[(1 + \rho^2/4)^2 - z^2]}{1 + \rho^2/4}, \frac{-z}{(1 + \rho^2/4)^2 - z^2}\right) \tag{4.10b}$$

Of the four solutions (4.7a), (4.7b) and (4.10a), (4.10b), the solution (4.10a) was given by Nagatomo (1989), while the others are new.

Example 2. Let the initial datum be

$$M_2(z; J) = \begin{pmatrix} 2z^2 + (1 - J^2)z + 1 & 2z^2 \\ 2z^2 & 2z^2 - (1 - J^2)z - J^2 \end{pmatrix} \tag{4.11}$$

Similar to Example 1, we obtain a double solution of equation (2.6) as

$$M_2(\rho, z; J) = \frac{1}{1 - (1 - J^2)\rho^2/2}$$
$$\times \begin{pmatrix} \frac{1}{4}(1 - J^2)\rho^4 - \rho^2 + 2z^2 + (1 - J^2)z + 1 & \frac{1}{4}(J^2 - 1)\rho^4 - \rho^2 + 2z^2 \\ \frac{1}{4}(J^2 - 1)\rho^4 - \rho^2 + 2z^2 & \frac{1}{4}(1 - J^2)\rho^4 - \rho^2 + 2z^2 - (1 - J^2)z - J^2 \end{pmatrix} \tag{4.12}$$

From equation (2.7), the associated double complex Ernst potential is

$$\mathscr{E}_2(J) = \frac{[1 - (1 - J^2)\rho^2/2] + J\cdot[\frac{1}{4}(J^2 - 1)\rho^4 - \rho^2 + 2z^2]}{\frac{1}{4}(1 - J^2)\rho^4 - \rho^2 + 2z^2 + (1 - J^2)z + 1} \tag{4.13}$$

Thus equations (2.3) give a pair of SAVGF solutions:

$$(f_2, \omega_2) = \left(\frac{2(1 - \rho^2)}{(\rho^2 - 1)^2 + (2z + 1)^2}, \frac{1}{2}\left[\rho^2 + \frac{\rho^2(2z - 1)^2}{1 - \rho^2}\right]\right). \tag{4.14a}$$

$$(\hat{f}_2, \hat{\omega}_2) = \left(\rho(2z^2 - \rho^2 + 1), \frac{2z^2 - \rho^2}{2z^2 - \rho^2 + 1}\right) \tag{4.14b}$$

Furthermore, under the transformation as in equation (4.8), $\mathscr{E}_2(J)$ is changed into $\tilde{\mathscr{E}}_2(J) = \mathscr{E}_2^{-1}(J)$, from which we obtain another pair of SAVGF solutions as

$$(f_2', \omega_2') = \left(\frac{2(1-\rho^2)}{(\rho^2-1)^2+(2z-1)^2}, \frac{1}{2}\left[\frac{\rho^2(2z+1)^2}{1-\rho^2} - \rho^2\right]\right) \tag{4.15a}$$

$$(\hat{f}_2', \hat{\omega}_2') = \left(\rho(\rho^2 - 2z + 1), \frac{2z^2-\rho^2}{\rho^2 - 2z^2 + 1}\right) \tag{4.15b}$$

The solution (4.15a) was given by Nagatomo (1989), while (4.14a), (4.14b), and (4.15b) are new solutions.

We can also write out some other initial datum $M(z; J)$. For example, we take, more generally,

$$M_3(z; J) = \begin{pmatrix} 1 & g(z; J) \\ g(z; J) & g^2(z; J) - J^2 \end{pmatrix} \tag{4.16}$$

$$M_4(z; J) = \begin{pmatrix} 2h^2(z; J) + (1 - J^2)h(z; J) + 1 & 2h^2(z; J) \\ 2h^2(z; J) & 2h^2(z; J) - (1 - J^2)h(z; J) - J^2 \end{pmatrix} \tag{4.17}$$

etc., where $g(z; J)$ and $h(z; J)$ are arbitrary double real polynomials of z. The matrices (4.16) and (4.17) both satisfy the condition (3.2). Once a concrete $g(z; J)$ or $h(z; J)$ is taken, we can obtain four SAVGF solutions. The initial data (4.3) and (4.11) are, respectively, special cases of (4.16) and (4.17) if we take $g(z; J) = h(z; J) = z$.

Example 3. We take another class of $M(z; J)$ as

$$M_5(z) = \begin{pmatrix} u(z) & 1 \\ 1 & 0 \end{pmatrix} \tag{4.18}$$

where $u(z)$ is an arbitrary real polynomial of z. Noting that $M_5(z)$ satisfies the condition (3.2) only when $J = \epsilon$, from Theorem 1 the solutions corresponding to the initial data in the form (4.18) satisfy (2.6) only when $J = \epsilon$ also. Since $[M_5(z)]_{22} = 0$, the initial datum (4.18) has no double dual (corresponds to $J = i$) matrix. If we choose $u(z) = z$, then, following procedures similar to Examples 1 and 2, we find

$$M_5(\rho, z) = \begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix} \tag{4.19}$$

This yields an SAVGF solution

$$(\hat{f}_5, \hat{\omega}_5) = \left(\rho z, \frac{1}{z}\right) \tag{4.20}$$

If we take $u(z) = z^2$, then we obtain

$$M_5'(\rho, z) = \begin{pmatrix} z^2 - \frac{1}{2}\rho^2 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.21}$$

The corresponding SAVGF solution is

$$(\hat{f}_5', \hat{\omega}_5') = \left(\rho\left(z^2 - \frac{1}{2}\rho^2\right), \frac{1}{z^2 - \frac{1}{2}\rho^2}\right) \tag{4.22}$$

etc. However, the solutions (4.19) and (4.21) in fact belong to the class of solutions given by van Stockum (1937).

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